A Remark on Long Range Scattering for the nonlinear Klein-Gordon equation

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1 Introduction

We consider the problem of scattering for the critical nonlinear Klein-Gordon in one space dimension:

$$\Box v + v = -\beta v^3, \tag{1.1}$$

where $\Box = \partial_t^2 - \partial_x^2$. Recall first that a solution of the linear Klein-Gordon, i.e. $\beta = 0$, is asymptotically given by (as ϱ tends to infinity)

$$u(t,x) \sim \varrho^{-1/2} e^{i\varrho}\, a(x/\varrho) + \varrho^{-1/2} e^{-i\varrho}\, \overline{a(x/\varrho)}, \qquad \text{where} \quad \varrho = (t^2 - |x|^2)^{1/2} \geq 0. \tag{1.2}$$

Here $a(x/\varrho) = (t/\varrho)\widehat{u}_+(-x/\varrho) = \sqrt{1 + x^2/\varrho^2}\,\widehat{u}_+(-x/\varrho)$, where $\widehat{u}_+(\xi) = \int u_+(x)\, e^{-ix\xi}\, dx$ denotes the Fourier transform with respect to x only, $\widehat{u}_+ = (\widehat{u}_0 - i(|\xi|^2 + 1)^{-1/2}\widehat{u}_1)/2$, where $u_0 = u\big|_{t=0}$ and $u_1 = \partial_t u\big|_{t=0}$. Here the right hand side is to be interpreted as 0 outside the light cone, when $|x| > t$. (1.2) can be proven using stationary phase, see e.g. [H1], where a complete asymptotic expansion into negative powers of ϱ was given. Recently, Delort[D1] proved that (1.1) with small initial data have a global solution with asymptotics of the form

$$v(t,x) \sim \varrho^{-1/2} e^{i\phi_0(\varrho, x/\varrho)}\, a(x/\varrho) + \varrho^{-1/2} e^{-i\phi_0(\varrho, x/\varrho)}\, \overline{a(x/\varrho)}, \qquad \phi_0(\varrho, x/\varrho) = \varrho + \frac{3}{8}\beta |a(x/\varrho)|^2 \ln \varrho \tag{1.3}$$

We consider the inverse problem of scattering, i.e. we show that for any given asymptotic expansion of the above form (1.3) there is a solution agreeing with it at infinity. More precisely, we show:

Theorem 1.1. *Suppose that a and $b_1 = b - b_0$ are fast decaying smooth real valued functions, where b_0 is a constant and $|a^{(k)}(x/\varrho)| + |b_1^{(k)}(x/\varrho)| \leq C_{Nkj}(1 + |x/\varrho|)^{-N}$, for any $k \geq 0$ and N. Let*

$$v_0(t,x) = \varrho^{-1/2}\, a(x/\varrho) \cos\phi(\varrho, x/\varrho) \qquad \phi(\varrho, x/\varrho) = \varrho + \frac{3}{8}\beta\, |a(x/\varrho)|^2 \ln \varrho + b(x/\varrho), \tag{1.4}$$

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interpreted as 0 *when* $|x| \geq t$. *Then there is* $T < \infty$ *such that* (1.1) *has a smooth solution* v *for* $T \leq t < \infty$ *satisfying* $v \sim v_0$ *as* $t \to \infty$. *More precisely for* $t \geq T$ *we have*

$$\|(v - v_0)(t, \cdot)\|_{L^\infty} + \sum_{|\alpha| \leq 1} \|\partial_{t,x}^\alpha (v - v_0)(t, \cdot)\|_{L^2} \leq C(1 + \ln(1+t))^2 (1+t)^{-1}. \tag{1.5}$$

Remark 1.2. Note that the initial data (a, b), as well as the parameter β, can be arbitrarily large.

Remark 1.3. We also get a complete asymptotic expansion, see Theorem 1.7.

Remark 1.4. Note that by (1.4)-(1.5) the solution constructed has bounded energy

$$\frac{1}{2} \int v_t(t, x)^2 + v_x(t, x)^2 + v(t, x)^2 \, dx + \frac{\beta}{4} \int v(t, x)^4 \, dx$$

Since the energy is conserved we get a global bound for each term in the energy if $\beta \geq 0$ or if β is sufficiently small and it follows that in this case the solution constructed in the theorem can be extended to a global solution for $-\infty < t < \infty$.

For the proof we start by introducing the hyperbolic coordinates

$$\varrho^2 = t^2 - x^2, \quad t = \varrho \cosh y, \quad x = \varrho \sinh y,$$

or

$$e^{2|y|} = \frac{t + |x|}{t - |x|}, \qquad \varrho^2 = t^2 - x^2 \tag{1.6}$$

Then

$$\Box + 1 = \partial_\varrho^2 - \varrho^{-2} \partial_y^2 + \varrho^{-1} \partial_\varrho + 1$$

and with

$$v(t, x) = \varrho^{-1/2} V(\varrho, y)$$

we get

$$(\Box + 1) v(t, x) = \varrho^{-1/2} \Big(\partial_\varrho^2 + 1 - \varrho^{-2} (\partial_y^2 - \tfrac{1}{4}) \Big) V(\varrho, y).$$

Hence in these coordinates (1.1) becomes the following equation for $V = \varrho^{1/2} v$:

$$\Psi(V) \equiv \partial_\varrho^2 V + \Big(1 + \frac{\beta}{\varrho} V^2 + \frac{1}{4\varrho^2} \Big) V - \frac{1}{\varrho^2} \partial_y^2 V = 0 \tag{1.7}$$

We are therefore led to first studying the ODE

$$L(g) \equiv \ddot{g} + \Big(1 + \frac{\beta}{\varrho} g^2 + \frac{1}{4\varrho^2} \Big) g = 0 \tag{1.8}$$

We will prove in the next section:

Proposition 1.5. *For any constants* a *and* b *let*

$$g_0(\varrho) = a \cos \phi, \qquad \phi = \varrho + \delta \ln \varrho + b, \qquad \delta = \frac{3}{8} \beta a^2 \tag{1.9}$$

Then, if $\delta \geq 0$ *is sufficiently small, the ODE* (1.8) *has a solution* g *satisfying*

$$|\dot{g} - \dot{g}_0| + |g - g_0| \leq C \frac{|a|}{\varrho}, \qquad \varrho \geq 1. \tag{1.10}$$

Remark 1.6. The importance of the above proposition is that the ODE (1.8) determines the correct phase function ϕ in the ansatz for the solution of the PDE (1.7). The precise form of the logarithmic correction to the phase is due to the long range nature of the interaction.

Theorem 1.7. *Let v be the solution in Theorem 1.1 and let $V = \rho^{1/2}v$. Then for each $k \geq 1$ there is V_k of the form (N,I finite)*

$$a(y)\cos\phi + \sum_{n=0}^{N} \sum_{i\leq I,\, 1\leq j\leq k} \Big(a_{ijn}(y)\cos n\phi + b_{ijn}(y)\sin n\phi\Big)\frac{\ln^i \varrho}{\varrho^{\,j}}, \tag{1.11}$$

such that

$$|\Psi(V_k)| \leq \frac{C}{\varrho^{\,k+1}}, \qquad |V - V_k| \leq \frac{C}{\varrho^{\,k+1}}, \qquad \varrho \geq 1. \tag{1.12}$$

Furthermore, a_{ijn}, b_{ijn} are monomials in a and its derivatives, of at least order 1 .

2 The first order asymptotics and small data existence at infinity for the ODE

We want to solve the ODE (1.8), subject to a given behavior at infinity.

Lemma 2.1. *Let, for any $|a| \leq 1$,*

$$g_1(\varrho\,) = a\cos\phi + \frac{\delta}{12\varrho}a\cos 3\phi, \qquad \phi = \varrho\, + \delta\ln\varrho\,, \qquad \delta = \frac{3}{8}\beta a^2 \tag{2.1}$$

Then

$$|L(g_1)| \leq K\frac{|a|(1+\delta)^2}{\varrho^{\,2}}, \qquad \varrho \geq 1 \tag{2.2}$$

Proof. Using that $\cos^3\phi = \big(\cos 3\phi + 3\cos\phi\big)/4$ we have ($A = A(t)$)

$$L(A\cos\phi) = \Big((1-\dot\phi^2)A + \frac{3\beta}{4\varrho}A^3 + \ddot A + \frac{A}{4\varrho^{\,2}}\Big)\cos\phi - (\ddot\phi A + 2\dot\phi\dot A)\sin\phi + \frac{\beta}{4\varrho}A^3\cos 3\phi \tag{2.3}$$

We get, for $A(t) = a$

$$L\big(a\cos\phi\big) = \Big(-\frac{\delta^2}{\varrho^{\,2}} + \frac{1}{4\varrho^{\,2}}\Big)a\cos\phi + \frac{\delta}{\varrho^{\,2}}\,a\sin\phi + \frac{2\delta}{3\varrho}\,a\cos 3\phi \tag{2.4}$$

and

$$\begin{aligned} L'(0)\Big(-\frac{\cos 3\phi}{8\varrho}\Big) &= \frac{\cos 3\phi}{\varrho} + \Big(9(\dot\phi^2 - 1) + \frac{1}{\varrho} + \frac{1}{4\varrho^{\,2}}\Big)\frac{\cos 3\phi}{8\varrho} + \Big(\ddot\phi - \frac{\dot\phi}{\varrho}\Big)\frac{3\sin 3\phi}{8\varrho} \\ &= \frac{\cos 3\phi}{\varrho} + \frac{\cos 3\phi}{8\varrho^{\,2}}(1 + 18\delta + \frac{9\delta^2}{\varrho} + \frac{1}{4\varrho}) - \frac{3\,\sin 3\phi}{8\varrho^{\,2}}(1 + \frac{2\delta}{\varrho}) \end{aligned} \tag{2.5}$$

where $L'(0)$ is given by (2.6). Therefore,

$$L(g_1) = L(a\cos\phi) - \frac{8\delta a}{12} L'(0)\big(\frac{-\cos 3\phi}{8\varrho}\big) + \frac{\beta}{\varrho}\big(g_1^3 - a^3\cos^3\theta\big) = O\big(\frac{(1+\delta)^2|a|}{\varrho^2}\big)$$

□

The linearized operator of L around 0 is given by

$$L'(0)g = \ddot{g} + g + \frac{1}{4\varrho^2}g = F \tag{2.6}$$

The inverse to this operator with vanishing data at ∞ is given by

$$g(\varrho) = -\int_\varrho^\infty E_s(\varrho)F(s)\,ds, \tag{2.7}$$

where $E_s(\varrho)$ is the forward fundamental solution of (2.6), i.e. $E_s(\varrho)$ satisfies $L_0(E_s) = 0$ and $E_s(s) = 0$, $E_s'(s) = 1$. The solution of (2.6) satisfies

$$\frac{d}{d\varrho}\big(\dot{g}^2 + g^2\big)^{1/2} = \frac{\dot{g}}{\big(\dot{g}^2 + g^2\big)^{1/2}}\Big(F - \frac{g}{4\varrho^2}\Big) \geq -|F| - \frac{\big(\dot{g}^2 + g^2\big)^{1/2}}{8\varrho^2}. \tag{2.8}$$

Multiplying by the integrating factor $e^{-1/(8\varrho)}$ we see that

$$\big(\dot{g}(\varrho)^2 + g(\varrho)^2\big)^{1/2} \leq e^{1/(8\varrho)}\int_\varrho^\infty |F(s)|\,ds \leq 2\int_\varrho^\infty |F(s)|\,ds, \qquad \varrho \geq 1 \tag{2.9}$$

Hence (2.7) defines a solution of (2.6) with vanishing data at infinity if the integral above is convergent.

We have

$$L(g) - L(g_1) = L'(0)(g - g_1) + \frac{\beta}{\varrho}G(g_1, g - g_1)(g - g_1), \qquad \text{where} \quad G(g,h) = \big(3g^2 + 3gh + h^2\big) \tag{2.10}$$

Therefore, to solve (1.8) we now have to solve the equation

$$L'(0)(g - g_1) = -\frac{\beta}{\varrho}G(g_1, g - g_1)(g - g_1) - L(g_1)$$

This is done by iteration. We therefore define a sequence h_k :

$$L'(0)(h_{k+1}) = -\frac{\beta}{\varrho}G(g_1, h_k)h_k - L(g_1), \qquad k \geq 0, \qquad h_0 = 0,$$

where by Lemma 2.1 (equation 2.2)

$$|L(g_1)| \leq K\frac{|a|(1+\delta)^2}{\varrho^2} \tag{2.11}$$

We will inductively assume that

$$|h_k| \leq 4K\frac{|a|(1+\delta)^2}{\varrho} \tag{2.12}$$

Then

$$|G(g_1, h_k)| \le C'|a|^2(1+\delta)^4 \tag{2.13}$$

and by (2.9) we have for $\varrho \ge 1$,

$$|h_{k+1}| \le \int_\varrho^\infty 2\Big(4\beta C'|a|^2(1+\delta)^4 K\frac{|a|(1+\delta)^2}{s^2} + K\frac{|a|(1+\delta)^2}{s^2}\Big)\, ds \le 4K\frac{|a|(1+\delta)^2}{\varrho}$$

if $\delta \sim \beta a^2$ is sufficiently small. This shows that we have a bounded sequence h_k, and similarly looking at differences shows that it converges and hence we get a solution to the ODE.

3 The first order asymptotic and small data existence at infinity for the PDE

In this section we prove Theorem 1.1 in the case of small data, or equivalently small β. This result follows from the general proof in section 4 but we want to first give the proof in the simple situation were the complete asymptotic expansion is not needed and one can clearly see that existence for the PDE follows from existence for the ODE.

We now use Proposition 1.5 and Lemma 2.1 to postulate the following form for the ansatz of the leading behavior of the solution of (1.1):

$$v_1(t,x) = \varrho^{-1/2}V_1(\varrho, y),$$

where

$$V_1 = a(y)\cos\phi(\varrho, y) + \frac{\delta}{12\varrho}a(y)\cos 3\phi(\varrho, y), \qquad \phi(\varrho, y) = \varrho + \delta\ln\varrho + b(y), \qquad \delta = \frac{3}{8}\beta a^2. \tag{3.1}$$

Here a, b are smooth functions of y, such that a and $b_1 = b - b_0$, where b_0 is a constant, are decaying exponentially fast. Note that this ansatz is obtained from Lemma 2.1 by simply making the constants a, b dependent on y. Here we assume that for all N,

$$|D_y^k a(y)| + |D_y^k b_1(y)| \le C_N e^{-N|y|} \le C_N\Big(\frac{t-|x|}{t+|x|}\Big)^{N/2} \le C_N\frac{\varrho^N}{t^N} \tag{3.2}$$

where we used (1.6) for the second inequality and $|x| \le t$. Therefore, for any N,

$$\frac{|\, a^{(k)}|}{\varrho^N} \le \frac{C_N}{t^N} \tag{3.3}$$

With notation as in (1.8) and (1.7) we have

$$\Box v_1 + v_1 + \beta v_1^3 = \varrho^{-1/2}\Psi(V_1) = \varrho^{-1/2}L(V_1) - \varrho^{-1/2}\frac{1}{\varrho^2}\partial_y^2 V_1 = F_1 \tag{3.4}$$

where if we choose N sufficiently large

$$|F_1| \le C_N\frac{\big(1+\beta\ln|1+\varrho|\big)^2}{\varrho^{5/2}}\, e^{-N|y|} \le C\frac{\big(1+\beta\ln|1+t|\big)^2}{t^{5/2}} \tag{3.5}$$

since $e^{-2|y|} = (t-|x|)/(t+|x|)$ and $|x| \le t$.

We now estimate the correction to v_1: let v be the exact solution of (1.1). We have

$$(\Box + 1)(v - v_1) = \beta G(v_1, v - v_1)(v - v_1) - F_1, \qquad \text{where} \quad G(v,w) = \big(3v^2 + 3vw + w^2\big) \tag{3.6}$$

Let w be the solution of

$$\Box w + w = F$$

with vanishing data at infinity, i.e. w is defined by

$$w(t,x) = -\int_t^\infty \int E(t-s, x-y) F(s,y)\, dy ds$$

where E is the forward fundamental solution of $\Box + 1$. By the energy inequality

$$\|\partial w(t,\cdot)\|_{L^2} + \|w(t,\cdot)\|_{L^2} \le \int_t^\infty \|F(s,\cdot)\|_{L^2}\, ds. \tag{3.7}$$

Again, we solve for $w = v - v_1$ by iteration: Let w_k be defined by $w_0 = 0$ and

$$(\Box + 1) w_{k+1} = \beta G(v_1, w_k) w_k + F_1, \qquad k \ge 0. \tag{3.8}$$

Since F_1 is supported in $|x| \le t$ it follows from (3.5) that

$$\|F_1(t,x)\|_{L^2} \le \frac{K(1 + \beta \ln|1+t|)^2}{t^2}. \tag{3.9}$$

We will inductively assume that

$$\|\partial w_k(t,\cdot)\|_{L^2} + \|w_k(t,\cdot)\|_{L^2} \le \frac{4K(1 + \beta \ln|1+t|)^2}{t} \tag{3.10}$$

Since by Hölder's inequality

$$w^2 \le 2 \int |w||w_x|\, dx \le 2\|w\|_{L^2} \|\partial w\|_{L^2} \le \|\partial w\|_{L^2}^2 + \|w\|_{L^2}^2$$

we also get

$$\|w_k(t,\cdot)\|_{L^\infty} \le \frac{4K(1 + \beta \ln|1+t|)^2}{t}$$

Since also (see (3.1) and (3.3))

$$\|v_1(t,\cdot)\|_{L^\infty} \le \frac{2C_0}{t^{1/2}} \tag{3.11}$$

it follows that for $t \ge t_K$, where t_K depends on K only,

$$\|G(v_1, w_k)(t,\cdot)\|_{L^\infty} \le 3\|v_1\|_{L^\infty}^2 + 3\|v_1\|_{L^\infty} \|w_k\|_{L^\infty} + \|w_k\|_{L^\infty}^2 \le \frac{48 C_0^2}{t}, \qquad t \ge t_K \tag{3.12}$$

Hence by the energy inequality (3.7), and (3.8), (3.9), (3.11)

$$\|w_{k+1}(t,\cdot)\|_{L^2} \le \int_t^\infty \big(\beta 8 C_0^2 + 1\big) \frac{K(1 + \beta \ln|1+s|)^2}{s^2} ds \le \frac{2K(1 + \beta \ln|1+t|)^2}{t}, \qquad t \ge t_K'$$

if $\beta > 0$ is sufficiently small and t_K' is sufficiently large. Estimating $\|\partial w_{k+1}\|_{L^2}$ in the same way, we conclude that (3.10) follows also for $k+1$.

4 Higher order asymptotics and existence for large data at infinity

Let us also consider the linearized operator at $a\cos\phi$:

$$L_0(g) = L'(g_0)\,g = L'(a\cos\phi)g = \ddot{g} + (1 + 4^{-1}\varrho^{-2} + 8\delta\varrho^{-1}\cos^2\phi)g$$

Lemma 4.1. *Suppose that $k \geq 1$. We have*

$$L_0\Big(\frac{\cos n\phi}{\varrho^{k}}\ln^i\!\varrho\Big) = (1-n^2)\frac{\cos n\phi}{\varrho^{k}}\ln^i\!\varrho + \sum_{k'=k+1}^{k+2}\sum_{i'\leq i}\sum_{n'=n-2}^{n+2}\Big(a^{kin}_{k'i'n'}\frac{\cos n'\phi}{\varrho^{k'}} + b^{kin}_{k'i'n'}\frac{\sin n'\phi}{\varrho^{k'}}\Big)\ln^{i'}\!\varrho \tag{4.1}$$

$$L_0\Big(\frac{\sin n\phi}{\varrho^{k}}\ln^i\!\varrho\Big) = (1-n^2)\frac{\sin n\phi}{\varrho^{k}}\ln^i\!\varrho + \sum_{k'=k+1}^{k+2}\sum_{i'\leq i}\sum_{n'=n-2}^{n+2}\Big(c^{kin}_{k'i'n'}\frac{\cos n'\phi}{\varrho^{k'}} + d^{kin}_{k'i'n'}\frac{\sin n'\phi}{\varrho^{k'}}\Big)\ln^{i'}\!\varrho \tag{4.2}$$

and

(4.3)
$$L_0\Big(\frac{\cos\phi}{\varrho^{k}}\ln^i\!\varrho\Big) = \Big(2k\frac{\sin\phi}{\varrho^{k+1}} + 4\delta\frac{\cos\phi}{\varrho^{k+1}} + 2\delta\frac{\cos 3\phi}{\varrho^{k+1}}\Big)\ln^i\!\varrho - 2i\frac{\sin\phi}{\varrho^{k+1}}\ln^{i-1}\!\varrho + \sum_{i'\leq i}\Big(a_{i'}\frac{\cos\phi}{\varrho^{k+2}} + b_{i'}\frac{\sin\phi}{\varrho^{k+2}}\Big)\ln^{i'}\!\varrho$$

(4.4)
$$L_0\Big(\frac{\sin\phi}{\varrho^{k}}\ln^i\!\varrho\Big) = \Big(-2k\frac{\cos\phi}{\varrho^{k+1}} + 2\delta\frac{\sin 3\phi}{\varrho^{k+1}}\Big)\ln^i\!\varrho + 2i\frac{\cos\phi}{\varrho^{k+1}}\ln^{i-1}\!\varrho + \sum_{i'\leq i}\Big(c_{i'}\frac{\cos\phi}{\varrho^{k+2}} + d_{i'}\frac{\sin\phi}{\varrho^{k+2}}\Big)\ln^{i'}\!\varrho$$

Proof. Since $\phi = \varrho + \delta\ln\varrho + b$ it follows that

$$\frac{d}{d\varrho}\Big(\frac{e^{in\phi}}{\varrho^{k}}\Big) = \frac{d}{d\varrho}\big(e^{in\varrho}\,e^{(i\delta n-k)\ln\varrho}\,e^{ibn}\big) = \Big(in + \frac{(n\delta i-k)}{\varrho}\Big)\frac{e^{in\phi}}{\varrho^{k}} \tag{4.5}$$

$$\frac{d^2}{d\varrho^2}\Big(\frac{e^{in\phi}}{\varrho^{k}}\Big) = \frac{d^2}{d\varrho^2}\big(e^{in\varrho}\,e^{(i\delta n-k)\ln\varrho}\,e^{ibn}\big) = \Big(-n^2 + \frac{2ni(n\delta i-k)}{\varrho} + \frac{c_{nk}}{\varrho^2}\Big)\frac{e^{in\phi}}{\varrho^{k}} \tag{4.6}$$

Hence

$$\frac{d^2}{d\varrho^2}\Big(\ln^i\!\varrho\,\frac{e^{in\phi}}{\varrho^{k}}\Big) = \Big(-n^2 + \frac{2ni(n\delta i-k)}{\varrho} + \frac{c_{nk}}{\varrho^2}\Big)\frac{e^{in\phi}}{\varrho^{k}}\ln^i\!\varrho + \sum_{k'=k+1}^{k+2}\sum_{i'=i-2}^{i-1} c_{kik'i'}\frac{e^{in\phi}}{\varrho^{k'}}\ln^{i'}\varrho \tag{4.7}$$

$$\frac{d^2}{d\rho^2}\Big(\frac{\cos n\phi}{\varrho^{k}}\Big) = -n^2\frac{\cos n\phi}{\varrho^{k}} - 2\delta n^2\frac{\cos n\phi}{\varrho^{k+1}} + 2kn\frac{\sin n\phi}{\varrho^{k+1}} + a_{kn}\frac{\cos n\phi}{\varrho^{k+2}} + b_{kn}\frac{\sin n\phi}{\varrho^{k+2}} \tag{4.8}$$

$$\frac{d^2}{d\rho^2}\Big(\frac{\sin n\phi}{\varrho^{k}}\Big) = -n^2\frac{\sin n\phi}{\varrho^{k}} - 2\delta n^2\frac{\sin n\phi}{\varrho^{k+1}} - 2kn\frac{\cos n\phi}{\varrho^{k+1}} + c_{kn}\frac{\cos n\phi}{\varrho^{k+2}} + d_{kn}\frac{\sin n\phi}{\varrho^{k+2}} \tag{4.9}$$

Since $\cos^2\phi\cos\phi = \big(3\cos\phi + \cos 3\phi\big)/4$ and $\cos^2\phi\sin\phi = \big(\sin\phi + \sin 3\phi\big)/4$, we have

$$\Big(1 + 8\delta\frac{\cos^2\phi}{\varrho}\Big)\frac{\cos\phi}{\varrho^{k}} = \frac{\cos\phi}{\varrho^{k}} + 6\delta\frac{\cos\phi}{\varrho^{k+1}} + 2\delta\frac{\cos 3\phi}{\varrho^{k+1}} \tag{4.10}$$

$$\Big(1 + 8\delta\frac{\cos^2\phi}{\varrho}\Big)\frac{\sin\phi}{\varrho^{k}} = \frac{\sin\phi}{\varrho^{k}} + 2\delta\frac{\sin\phi}{\varrho^{k+1}} + 2\delta\frac{\sin 3\phi}{\varrho^{k+1}} \tag{4.11}$$

□

Definition 4.2. *Let $\mathcal{S}_k$ denote the family of finite sums (N,I,j-sum finite) of the form*

$$\sum_{n=0}^{N} \sum_{i \le I,\, j \ge k} \Big(a_{ijn}(y) \cos n\phi + b_{ijn}(y) \sin n\phi \Big) \frac{\ln^i \varrho}{\varrho^j}, \qquad \phi = \varrho + \delta \ln \varrho\,, \quad \delta = \frac{3}{8}\beta a(y)^2 \tag{4.12}$$

where for any N and ℓ there is a constant such that

$$|D_y^\ell a_{ijk}(y)| + |D_y^\ell b_{ijk}(y)| \le C_{N\ell}\, e^{-N|y|} \tag{4.13}$$

Furthermore, let $\mathring{\mathcal{S}}_k$ denote the family of finite sums of the above form but with

$$a_{ik1} = b_{ik1} = 0, \qquad \textit{for all} \quad i$$

Lemma 4.3. *If $k \ge 1$ and $\mathring{\Sigma}_k \in \mathring{\mathcal{S}}_k$, then there are $\Sigma_k \in \mathcal{S}_k$ and $\mathring{\Sigma}_{k+1} \in \mathring{\mathcal{S}}_{k+1}$ such that*

$$L_0 \Sigma_k = \mathring{\Sigma}_k + \mathring{\Sigma}_{k+1} \tag{4.14}$$

Proof. First we use the first part of the previous lemma to invert the terms with $k' = k$ and $n \ne 1$. Then we use the second part of the previous lemma to successively remove the terms with $n = 1$ by lowering the logarithms. First note that an element of $\mathring{\mathcal{S}}_k$ can be written as a

$$\sum_{n \ne 1} \sum_{k' \ge k,\, i} \big(\alpha_{ik'n} \frac{\cos n\phi}{\varrho^{k'}} + \beta_{ik'n} \frac{\sin n\phi}{\varrho^{k'}}\big) \ln^i(\varrho) + \sum_{k' \ge k+1,\, i} \big(\alpha'_{ik'} \frac{\cos\phi}{\varrho^{k'}} + \beta'_{ik'} \frac{\sin\phi}{\varrho^{k'}}\big) \ln^i(\varrho) = I_{nr} + I_{res}$$

The sum over $k' \ge k+1$ is due to the fact that $\mathring{\Sigma}_k$ unlike Σ_k is "nonresonant", that is do not contain lowest order terms in ϱ for $n = 1$. (see definition of the space $\mathring{S}_k$). Now, given such element $\mathring{\Sigma}_k$, we use (4.1), (4.2) to obtain

$$L_0 \big(\sum_{n \ne 1} \frac{1}{1-n^2} \sum_{k' \ge k,\, i} \big(\alpha_{ik'n} \cos n\phi + \beta_{ik'n} \sin n\phi\big) \frac{\ln^i \varrho}{\varrho^{k'}} \big) = I_{nr} + \mathring{\Sigma}_{k+1}$$

We are therefore left with inverting L_0 on I_{res}. To this end we use (4.3),(4.4), to obtain :

$$\begin{aligned} L_0 \big(\frac{1}{2k} \frac{\cos\phi}{\varrho^k} \ln^i \varrho + \frac{4\delta}{2k} \frac{\sin\phi}{\varrho^k} \ln^i \varrho \big) &= \frac{\sin\phi}{\varrho^{k+1}} \ln^i \varrho + \\ O(\varrho^{-k-1} \ln^i \varrho)(\sin 3\phi, \cos 3\phi) + O(\varrho^{-k-1} &\ln^{i-1} \varrho)(\sin\phi, \cos\phi) + O(\varrho^{-k-2} \ln^i \varrho)(\sin\phi, \cos\phi) \end{aligned} \tag{4.15}$$

and similar formula for $\cos\phi\, \varrho^{-k-1} \ln^i \varrho$. (Here $(f,g) \equiv \alpha a + \beta g$ for some numbers α, β.) Hence, we can invert L_0 on $(\cos\phi, \sin\phi)\, \varrho^{-k-1} \ln^i \varrho$ up to nonresonant terms in $\mathring{S}_{k+1}$, S_{k+2} and resonant terms in S_{k+1} but with one less power of $\ln \varrho$. Hence, by iteration, eliminate all such terms, in each step one less power of $\ln \varrho$. $\square$

We must then show that the products of the above classes are properly mapped as well as the Laplacian acting on the above classes. We want to solve

$$\Psi(V) = \partial_\varrho^2 V + \Big(1 + \frac{\beta}{\varrho} V^2 + \frac{1}{4\varrho^2}\Big) V - \varrho^{-2} \partial_y^2 V = 0 \tag{4.16}$$

by iteration, starting from

$$V_0 = a\cos\phi, \qquad \text{where} \quad \phi = \varrho + 3\beta 8^{-1} a^2 \ln\varrho + b \tag{4.17}$$

and $a = a(y)$, $b = b(y)$. We have:

Lemma 4.4. *There is a sequence V_k, $k = 0, ...,$ such $V_k - V_0 \in \mathcal{S}_1$, $\Psi(V_k) \in \mathring{\mathcal{S}}_{k+1}$ and $V_k - V_{k-1} \in \mathcal{S}_k$.*

Proof. We have

$$\partial_y^2\big(a\cos\phi\big) = \sum_{j=0}^{2}\big(a_j\cos\phi + b_j\sin\phi\big)\ln^j\varrho \tag{4.18}$$

for some functions $a_j(y)$, and $b_j(y)$ which are at least linear in $a^{(k)}, b^{(k)}$, $(0 \le k \le 2)$. It follows that

$$\Psi(V_0) = -\frac{\delta^2}{\varrho^2}a\cos\phi + \frac{\delta}{\varrho^2}a\sin\phi + \frac{2\delta}{3\varrho}a\cos 3\phi + \frac{a\cos\phi}{4\varrho^2} - \frac{1}{\varrho^2}\partial_y^2\big(a\cos\phi\big) \in \mathring{\mathcal{S}}_1 \tag{4.19}$$

This proves the lemma for $k = 0$ and in what follows we will assume the lemma for k replaced by $k-1$ and show that this implies the lemma also for k.

We have

$$\Psi'(V)W = \partial_\varrho^2 W + \Big(1 + 3\frac{\beta}{\varrho}V^2 + \frac{1}{4\varrho^2}\Big)W - \varrho^{-2}\partial_y^2 W \tag{4.20}$$

Since the operator $\varrho^{-2}\partial_y^2\psi$ maps $\mathcal{S}_k \to \mathcal{S}_{k+2} \subset \mathring{\mathcal{S}}_{k+1}$ it follows that

$$\Psi'(V_0) = L_0 - \varrho^{-2}\partial_y^2 V \tag{4.21}$$

can be inverted in the same spaces as L_0 in Lemma 4.3, i.e. if $k \ge 1$ and $\mathring{\Sigma}_k \in \mathring{\mathcal{S}}_k$, then there are $\Sigma_k \in \mathcal{S}_k$ and $\mathring{\Sigma}_{k+1} \in \mathring{\mathcal{S}}_{k+1}$ such that

$$\Psi'(a\cos\phi)\Sigma_k = \mathring{\Sigma}_k + \mathring{\Sigma}_{k+1} \tag{4.22}$$

Moreover if $V_n - V_0 \in \mathcal{S}_1$ it follows that $\big(\Psi'(V_n) - \Psi'(V_0)\big)\Sigma_k = 3\beta(V_n^2 - V_0^2)\Sigma_k/\varrho \in \mathcal{S}_{k+2} \in \mathring{\mathcal{S}}_{k+1}$ so $\Psi'(V_n)$ also satisfies

$$\Psi'(V_n)\Sigma_k = \mathring{\Sigma}_k + \mathring{\Sigma}_{k+1} \tag{4.23}$$

for some other $\mathring{\Sigma}_{k+1}$.

Given V_{k-1} such that $\Psi(V_{k-1}) \in \mathring{\mathcal{S}}_k$ and $V_{k-1} - V_0 \in \mathcal{S}_1$ we now find V_k such that $V_k - V_{k-1} \in \mathcal{S}_k$ by solving

$$\Psi'(V_{k-1})(V_k - V_{k-1}) + \Psi(V_{k-1}) \in \mathring{\mathcal{S}}_{k+1} \tag{4.24}$$

which is possible, by (4.23). Then with $\Phi(V, U) = 3V + U$

$$\Psi(V_k) = \Psi(V_{k-1}) + \Psi'(V_{k-1})(V_k - V_{k-1}) + \frac{\beta}{\varrho}\Phi(V_{k-1}, V_k - V_{k-1})(V_k - V_{k-1})^2 \in \mathring{\mathcal{S}}_{k+1} \tag{4.25}$$

$\square$

We have now found v_N, for any N , such that

$$\Box v_N + v_N + \beta v_N^3 = F_N = O(t^{-N-5/2}), \qquad v_N - v_0 = O(t^{-3/2}\ln t)$$

It follows that there is a constant $C_0 < \infty$ independent of N and another constant $t_N < \infty$ depending on N such that

$$|v_N| \le 2C_0 t^{-1/2}, \qquad t \ge t_N$$

We then define $w_0 = 0$ and for $l \ge 1$:

$$(\Box + 1)w_{l+1} = \beta G(v_N, w_l)w_l + F_N, \qquad l \ge 0.$$

Since F_N is supported in $|x| \le t$ it follows from (3.5) that

$$\|F_N(t,x)\|_{L^2} \le \frac{K_N}{t^{N+1}}$$

We will inductively (in l) assume that

$$\|\partial w_l(t,\cdot)\|_{L^2} + \|w_l(t,\cdot)\|_{L^2} \le \frac{4K_N}{Nt^N} \tag{4.26}$$

Since by Hölder's inequality

$$w^2 \le 2\int |w||w_x|\,dx \le 2\|w\|_{L^2}\|\partial w\|_{L^2} \le \|\partial w\|_{L^2}^2 + \|w\|_{L^2}^2$$

we also get

$$\|w_l(t,\cdot)\|_{L^\infty} \le \frac{4K_N}{N\,t^N}$$

Since also

$$\|v_N(t,\cdot)\|_{L^\infty} \le \frac{2C_0}{t^{1/2}}, \qquad t \ge t_N$$

where C_0 is independent of N, it follows that

$$\|G(v_N, w_l)(t,\cdot)\|_{L^\infty} \le \frac{8C_0}{t}, \qquad t \ge t_N'$$

Hence by the energy inequality

$$\|\partial w_{l+1}(t,\cdot)\|_{L^2} + \|w_{l+1}(t,\cdot)\|_{L^2} \le \int_t^\infty \frac{\beta 8C_0}{s}\frac{4K_N}{Ns^N}ds + \frac{K_N}{s^{N+1}}\,ds = \Big(\frac{32\,\beta C_0}{N} + 1\Big)\frac{K_N}{N\,t^N} \le \frac{2K_N}{Nt^N}, \qquad t \ge t_N''$$

if t_N'' and N are sufficiently large. Hence (4.26) follows also for $l+1$.

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